Math 10A HW8 Solutions

(1) False by definition

(2) False, consider $\int_{-\infty}^{+\infty} x\,dx$ where $f(x)=x$ is an odd function but the integral diverges

(3) True, application of Comparison theorem

(4)

(a) $\int_{5}^{\infty} \frac{1}{x^3}dx = \lim_{t\to\infty} \int_{5}^{t} \frac{1}{x^3}dx$

$= \lim_{t\to\infty} \left(-\frac{1}{2x^2}\right)\Big|_5^t = \lim_{t\to\infty} \left(-\frac{1}{2t^2} + \frac{1}{50}\right)$

$= \frac{1}{50}$

(b) $\int_{0}^{\infty} e^{-x}dx = \lim_{t\to\infty} \int_{0}^{t} e^{-x}dx$

$= \lim_{t\to\infty} (e^{-x})\Big|_0^t = \lim_{t\to\infty} \left(\frac{1}{e^t} - \frac{1}{e^0}\right)$

$= -1$

(c) $\int_{\pi}^{\infty} \cos(x)dx = \lim_{t\to\infty} \int_{\pi}^{t} \cos(x)dx$

$= \lim_{t\to\infty} \left(\cos t - \cos(\pi)\right)$ diverges!

(d) $\int_{-\infty}^{1} \frac{1}{x}\,dx = \lim_{t \to -\infty} \int_{t}^{1} \frac{1}{x}\,dx$

$= \lim_{t \to -\infty} (\ln x)\Big|_t^1 = \lim_{t \to -\infty} (\ln 1 - \ln t)$

$= \lim_{t \to -\infty} (-\ln t)$ diverges

(e) $\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{+\infty} \frac{1}{1+x^2}\,dx$

$\Rightarrow \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx = \lim_{t \to -\infty} (\tan^{-1} x)\Big|_t^0$

$= \lim_{t \to -\infty} (\tan^{-1} 0 - \tan^{-1} t) = \frac{\pi}{2}$

$\Rightarrow \int_{0}^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{t \to \infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx = \lim_{t \to \infty} (\tan^{-1} x)\Big|_0^t$

$= \pi/2$

So $\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \pi/2 + \pi/2 = \pi.$

(5) $\displaystyle\int_0^{\infty} \frac{1}{1+e^x}\,dx$ converges !

Notice $\dfrac{1}{1+x^2} \geq \dfrac{1}{1+e^x}$ for $x \geq 0$

and we've computed $\displaystyle\int_0^{\infty} \frac{1}{1+x^2}\,dx$ in 4e (converges).

(6) False, notice degree of numerator & denominator are the same so we must perform (polynomial) long division 1st.

(7)

(a) $\displaystyle\int \frac{10}{(x+1)^2(x-1)}\,dx$

$$\frac{10}{(x+1)^2(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-1)}$$

$$10 = A(x+1)(x-1) + B(x-1) + C(x+1)^2$$

$$\Rightarrow 10 = A(x^2-1) + Bx - B + C(x^2+2x+1)$$

$$10 = \cancel{Ax^2} - A + Bx - B + \cancel{Cx^2} + C2x + C$$

$$10 = (A+C)x^2 + (B+2C)x + (C-A-B)$$

$$\Rightarrow A+C=0 \rightarrow \boxed{A=-C = -5/2}$$

$$B+2C=0 \rightarrow \boxed{B=-2C = -5}$$

$$C-A-B=10 \rightarrow C = 10+A+B$$

$$C = 10 - C - 2C$$

$$4C = 10 \Rightarrow \boxed{C = \tfrac{10}{4} = \tfrac{5}{2}}$$

$$\int \frac{10}{(x+1)(x^2-1)}\,dx = \int \frac{\left(-\frac{5}{2}\right)}{x+1}\,dx + \int \frac{-5}{(x+1)^2}\,dx + \int \frac{\left(\frac{5}{2}\right)}{x-1}\,dx$$

$$= -\frac{5}{2}\ln(x+1) + \int \frac{-5}{(x+1)^2}\,dx + \frac{5}{2}\ln(x-1)$$

$$= -\frac{5}{2}\ln(x+1) + \frac{5}{x+1} + \frac{5}{2}\ln(x-1) + C$$

(b) $$\int \frac{2x+1}{x^2-5x+6}\,dx = \int \frac{2x+1}{(x-3)(x-2)}\,dx$$

$$\frac{2x+1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} = \frac{7}{x-3} + \frac{-5}{x-2}$$

$\Rightarrow$ $2x+1 = A(x-2) + B(x-3)$

$2x+1 = Ax - 2A + Bx - 3B$

$2x+1 = (A+B)x + (-2A-3B)$

$\Rightarrow$ $A+B=2$ so $A = 2-B = 2-(-5) = 7$

$-2A-3B=1$

$\Leftrightarrow$ $-2(2-B)-3B=1$

$-4+2B-3B=1$

$-4-B=1$

$-5 = -4-1 = B$

So $$\int \frac{2x+1}{(x-3)(x-2)}\,dx = \int \frac{7}{x-3}\,dx + \int \frac{-5}{x-2}\,dx$$

$$= 7\ln(x-3) - 5\ln(x-2) + C$$

(c) $\displaystyle\int \frac{x-1}{x^2+2x+1}\,dx = \int \frac{x-1}{(x+1)^2}\,dx$

$$\frac{x-1}{(x+1)^2} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} = \frac{1}{x+1} + \frac{-2}{(x+1)^2}$$

$x-1 = A(x+1)+B$

$\Rightarrow \; x-1 = Ax + (A+B)$

$A = 1$

$A+B = -1 \Rightarrow B = -1-A = -2$

So $\displaystyle\int \frac{x-1}{x^2+2x+1}\,dx = \int \frac{1}{x+1}\,dx - \int \frac{2}{(x+1)^2}\,dx$

$\displaystyle = \ln(x+1) + \frac{2}{x+1} + C$

(8) $$\frac{1}{(x+1)(x-1)(x^2+1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$